Descriptional Complexity under Parikh Equivalence

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Research in Descriptional Complexity

How succinctly a formal language can be described by a formalism in comparison with other formalisms.

Take the length of description as complexity measure.

- What is the blow-up when changing from one model to another?
- Are there languages such that a blow-up is achieved?

Investigation of *upper bounds* and *lower bounds*.

Standard equivalence: 1NFAs vs 1DFAs

Subset construction [Rabin&Scott '59]

1NFA
n states
L

$\Longrightarrow$

1DFA
2^n states
L

Moreover, this state bound cannot be reduced in the worst case
[Meyer&Fischer '71, Moore '71]

What happens if we do not care of the order of symbols in the strings?

This problem is related to the concept of *Parikh equivalence*
[Parikh '66]

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[Parikh '66]

Parikh equivalence: preliminaries

- $\Sigma = \{a_1, \ldots, a_m\}$ alphabet of m symbols
- $|w|_a$ be the number of occurrences of a in $w \in \Sigma^*$

Parikh map

The *Parikh map* $\psi : \Sigma^* \to \mathbb{N}^m$ associates with a word $w \in \Sigma^*$ the m-dimensional nonnegative vector $(|w|_{a_1}, |w|_{a_2}, \ldots, |w|_{a_m})$.

Parikh image

The *Parikh image of a language* L is $\psi(L) = \{\psi(w) \mid w \in L\}$.

- $w_1 =_\pi w_2$ if $\psi(w_1) = \psi(w_2)$
- $L_1 =_\pi L_2$ if $\psi(L_1) = \psi(L_2)$

Parikh equivalence: Parikh's theorem

Theorem ([Parikh '66])

The Parikh image of a context-free language is a semilinear set. Equivalently, for each context-free language $L \subseteq \Sigma^*$, there exists a Parikh equivalent regular language $R \subseteq \Sigma^*$.

Example ($L =_\pi R$)

$$L = \{a^n b^n \mid n \geq 0\} \quad \text{and} \quad R = (ab)^*$$

have the same Parikh image, namely the set

$$\{(n, n) \mid n \geq 0\}$$

Parikh equivalence

Motivations and current research

Interesting theoretical properties:

- wrt Parikh equivalence regular and context-free languages are indistinguishable [Parikh '66]
- Unary context-free languages are regular [Ginsburg&Rice '62]

Connections with number theory and logic:

- Semilinear sets [Ginsburg&Spanier '64, Ginsburg&Spanier '66]
- Presburger arithmetics [Presburger&Jabcquette '91, Presburger '91]
- Formal verification [Göller&Mayr&To '09]
- Petri nets [Esparza '97]

Our Goal

We investigate the conversion of 1NFAs and CFGs into *Parikh equivalent* 1DFAs and 2DFAs, from a *descriptional complexity* point of view.

Problem (1NFAs to 1DFAs and 2DFAs)

1NFA
n states

$\Longrightarrow_{\pi}$

1DFA, 2DFA
how many states?

Problem (CFGs to 1DFAs and 2DFAs)

CFG
in Chomsky normal form
h variables

$\Longrightarrow_{\pi}$

1DFA, 2DFA
how many states?

From 1NFAs to Parikh equivalent 1DFAs

Our first contribution:

Problem (1NFAs to 1DFAs)

$$\underset{\substack{n \text{ states} \\ L_1}}{1\text{NFA}} \Longrightarrow_{\pi} \underset{\substack{\text{how many states?} \\ L_2}}{1\text{DFA}}$$

- Upper bound: 2^n
  by subset construction [Rabin&Scott '59]
- Lower bound: $e^{\sqrt{n \cdot \ln n}}$
  This bound derives from the *unary case*:
  the state cost of the conversion of unary n-state 1NFAs
  into equivalent 1DFAs is $e^{\Theta(\sqrt{n \cdot \ln n})}$ [Chrobak '86]

Converting 1NFAs accepting only nonunary strings

A preliminary step:

Problem (1NFAs to 1DFAs, restricted)

1NFA *s.t. each accepted string is nonunary*
n states
L_1

$\Longrightarrow_{\pi}$

1DFA
how many states?
L_2

Quite surprisingly, we can obtain a 1DFA with a number of states *polynomial* in n,

i.e., this conversion is less expensive than the conversion in the unary case, which costs $e^{\Theta(\sqrt{n \cdot \ln n})}$

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Converting 1NFAs accepting only nonunary strings

The conversion uses a modification of the following result:

Theorem ([Kopczyński&To '10])

Given $\Sigma = \{a_1, \ldots, a_m\}$, there is a polynomial p s.t. for each n-state 1NFA A over Σ,

$$\psi(L(A)) = \bigcup_{i \in I} Z_i$$

where:

- *I is a set of at most $p(n)$ indices*
- *for $i \in I$, $Z_i \subseteq \mathbb{N}^m$ is a linear set of the form:*

$$Z_i = \{\alpha_0 + n_1\alpha_1 + \cdots + n_k\alpha_k \mid n_1, \ldots, n_k \in \mathbb{N}\}$$

with

  - $0 \leq k \leq m$
  - *the components of α_0 are bounded by $p(n)$*
  - *$\alpha_1, \ldots, \alpha_k$ are linearly independent vectors from $\{0, 1, \ldots, n\}^m$*

Converting 1NFAs accepting only nonunary strings

Outline: linear sets

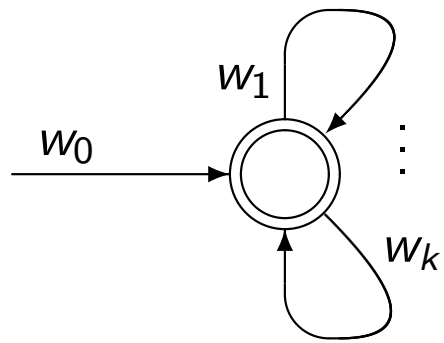
Each above linear set

$$Z_i = \{\alpha_0 + n_1\alpha_1 + \cdots + n_k\alpha_k \mid n_1, \ldots, n_k \in \mathbb{N}\}$$

can be converted into a poly size 1DFA accepting a language

$$R_i = w_0(w_1 + \cdots + w_k)^*$$

s.t. $\psi(w_j) = \alpha_j$, $j = 0, \ldots, k$, and
$w_1, \ldots, w_k$ *begin with different letters*



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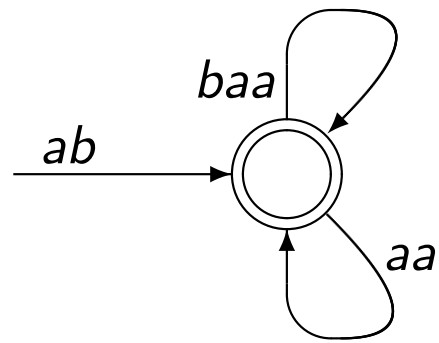
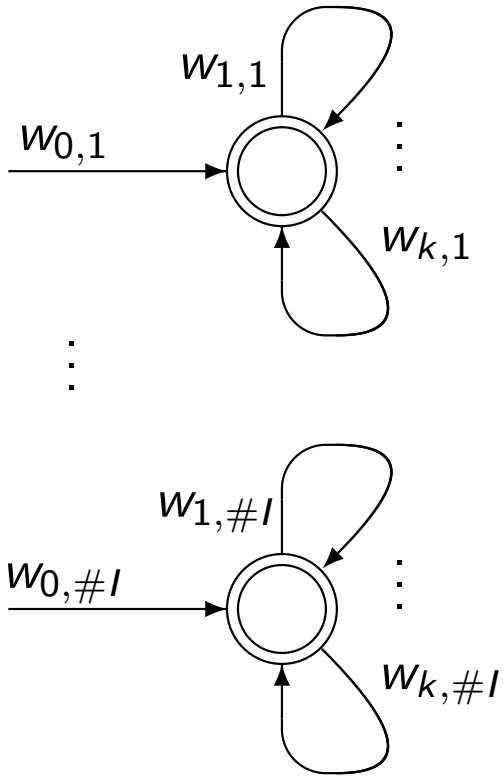
Example:

- $\{(1,1) + n_1(2,1) + n_2(2,0) \mid n_1, n_2 \geq 0\}$
- $ab(baa + aa)^*$



Converting 1NFAs accepting only nonunary strings

Outline: from linear to semilinear



- Standard construction for union of 1DFAs: number of states = *product*

  $\#I \leq p(n) \Rightarrow$ Too large!!!

- Strings $w_{0,i}$ can be replaced by Parikh equivalent strings $\hat{w}_{0,i}$ in such a way that $W_0 = \{\hat{w}_{0,i} \mid i \in I\}$ is a *prefix code*

- After this change:
  number of states $\leq$ *sum* Polynomial!!!

Converting 1NFAs accepting only nonunary strings

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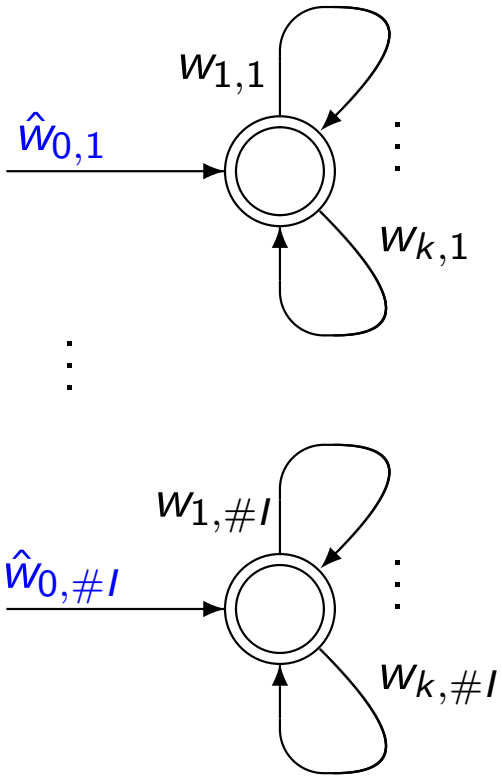

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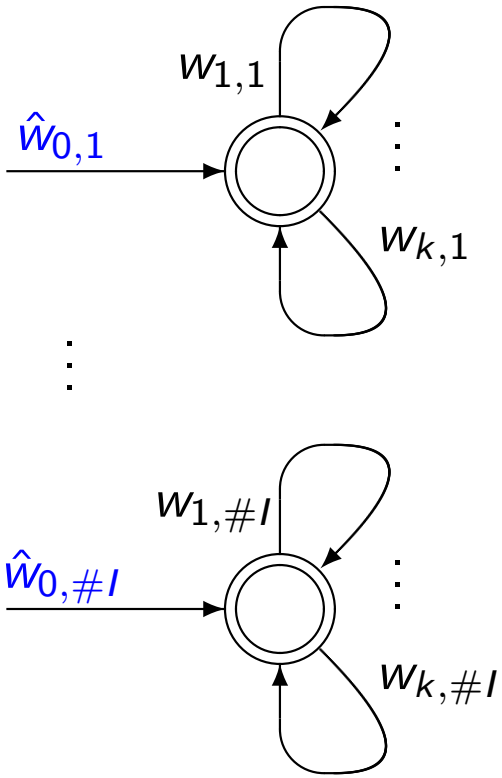

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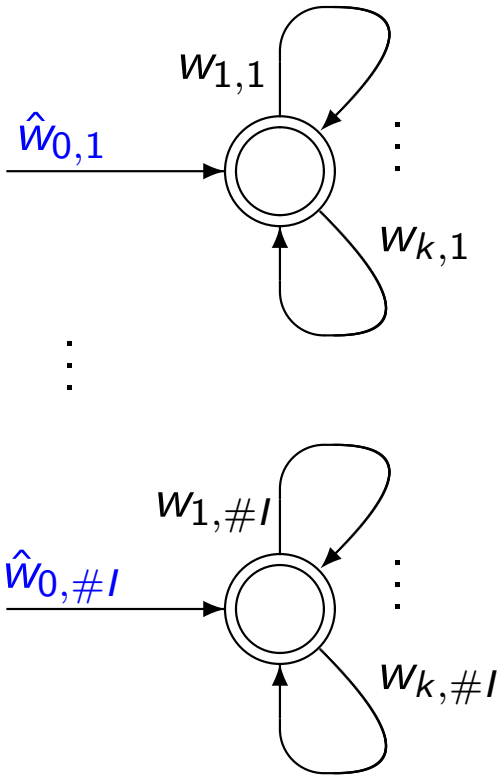

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- After this change:
  number of states $\leq$ *sum* Polynomial!!!

Theorem

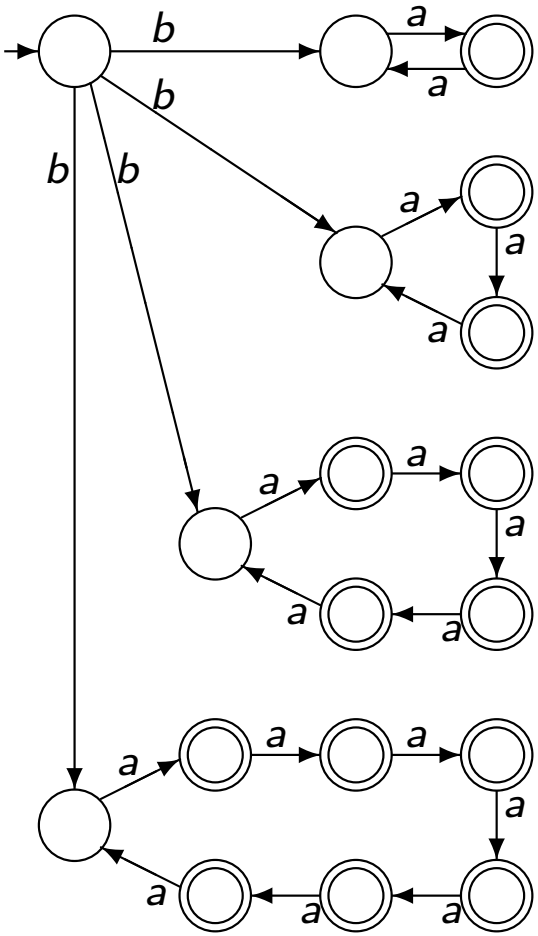
For each n-state 1NFA *accepting a language none of whose words are unary, there exists a Parikh equivalent* 1DFA *with a number of states polynomial in n.*

Converting 1NFAs accepting only nonunary strings

Example

Let us consider the following language

$$L = \{ba^n \mid n \bmod 210 \neq 0\}$$



- L does not contain any unary word.
- $L_1 = \{ba^n \mid n \bmod 2 \neq 0\}$,
- $L_2 = \{ba^n \mid n \bmod 3 \neq 0\}$,
- $L_3 = \{ba^n \mid n \bmod 5 \neq 0\}$,
- $L_4 = \{ba^n \mid n \bmod 7 \neq 0\}$,
- $L = L_1 \cup L_2 \cup L_3 \cup L_4$.

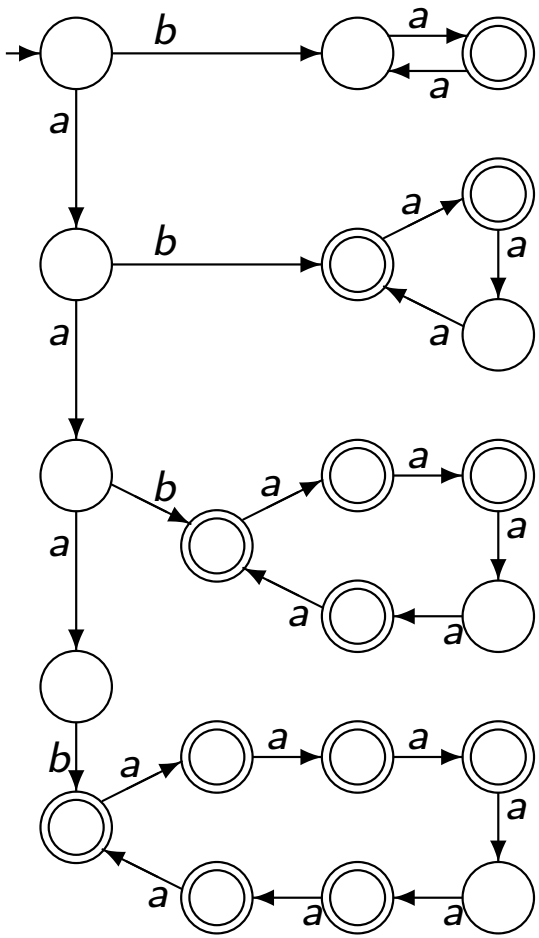
- L is accepted by the 18-state 1NFA A.
- The smallest 1DFA for L uses 211 states.

Converting 1NFAs accepting only nonunary strings

Example

We can build a complete 1DFA A' with only 22 states, accepting a language L' Parikh equivalent to L.

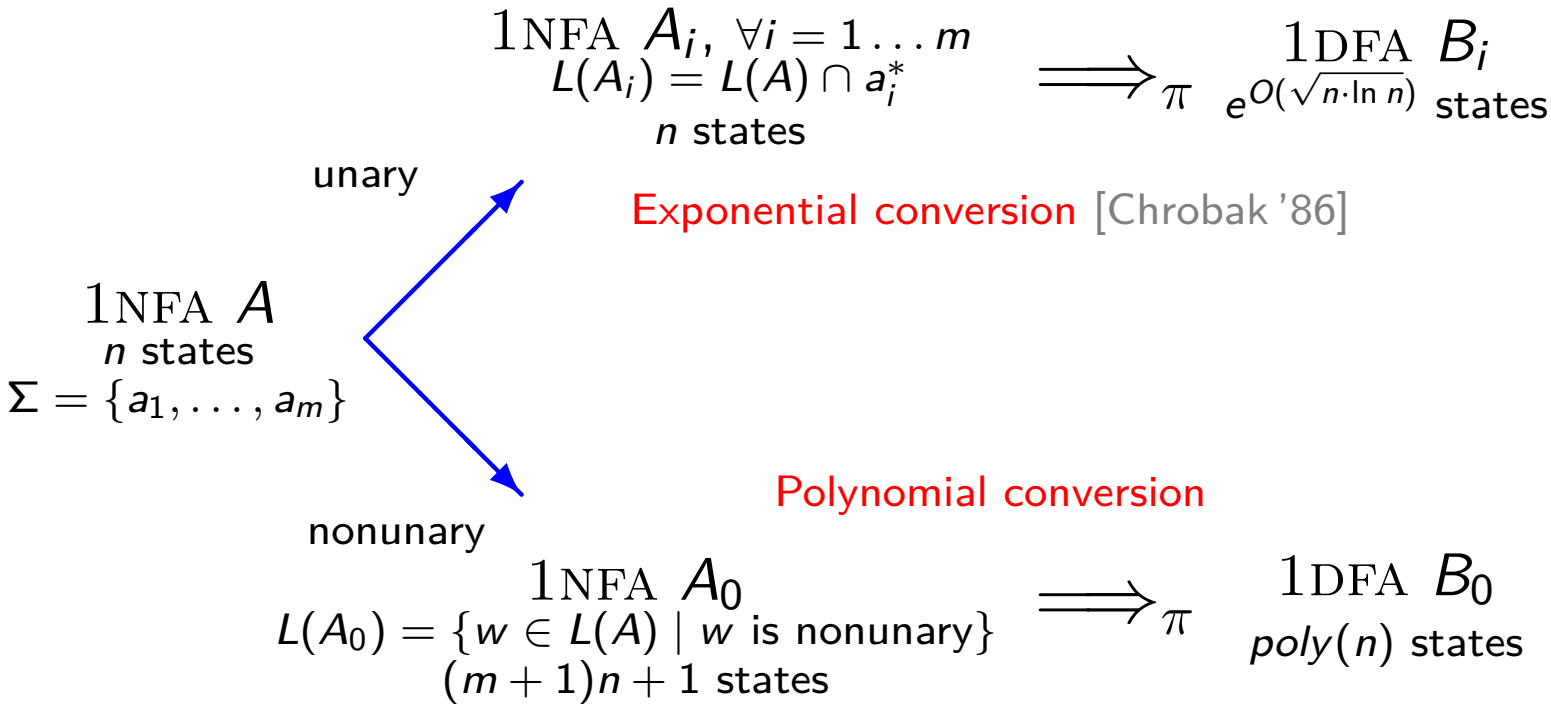
- $L_i =_\pi L'_i$, $i = 1, \ldots, 4$
- Words in L'_i begin with the prefix $a^{i-1}b$
- $L'_1 = \{ba^n \mid n \bmod 2 \neq 0\} = L_1$,
- $L'_2 = \{aba^{n-1} \mid n \bmod 3 \neq 0\}$,
- $L'_3 = \{a^2ba^{n-2} \mid n \bmod 5 \neq 0\}$,
- $L'_4 = \{a^3ba^{n-3} \mid n \bmod 7 \neq 0\}$,
- $L' = L'_1 \cup L'_2 \cup L'_3 \cup L'_4$.



From 1NFAs to Parikh equivalent 1DFAs: general case

1NFA A
n states
$\Sigma = \{a_1, \ldots, a_m\}$

From 1NFAs to Parikh equivalent 1DFAs: general case



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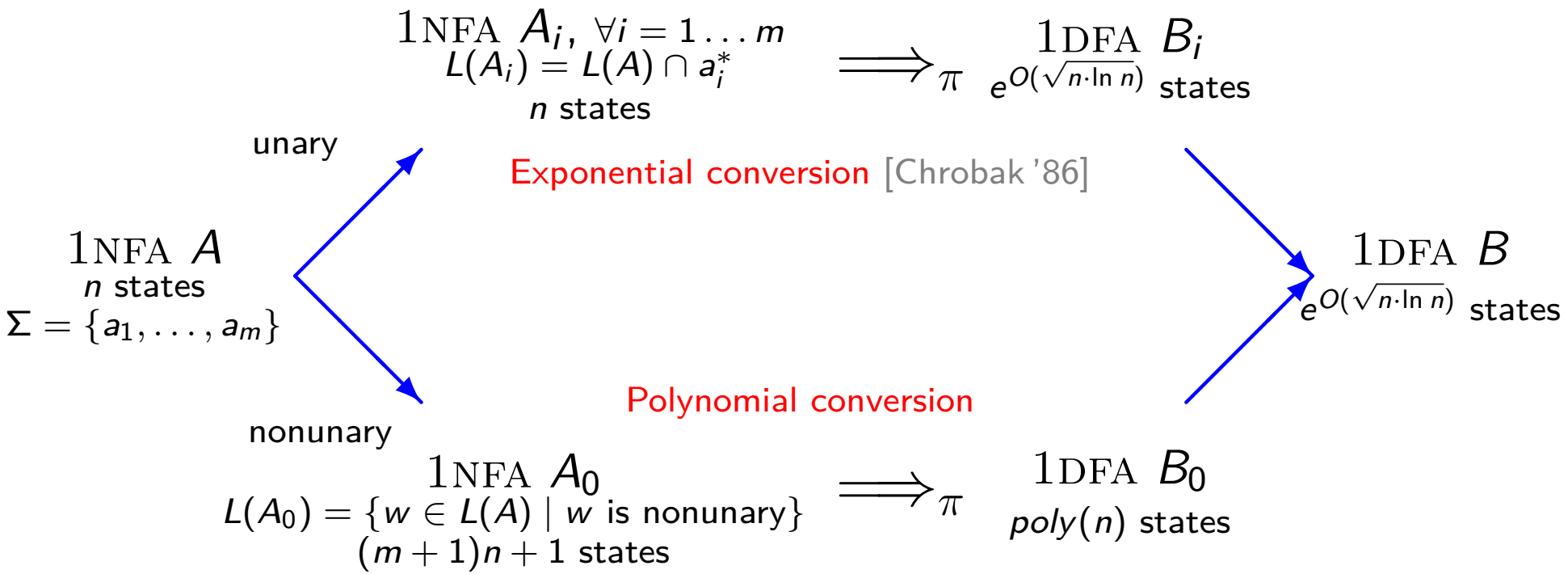


Theorem

For each n-state 1NFA *over* Σ, *there exists a Parikh equivalent* 1DFA *with* $e^{O(\sqrt{n \cdot \ln n})}$ *states. Furthermore, this cost is optimal.*

From CFGs to Parikh equivalent 1DFAs

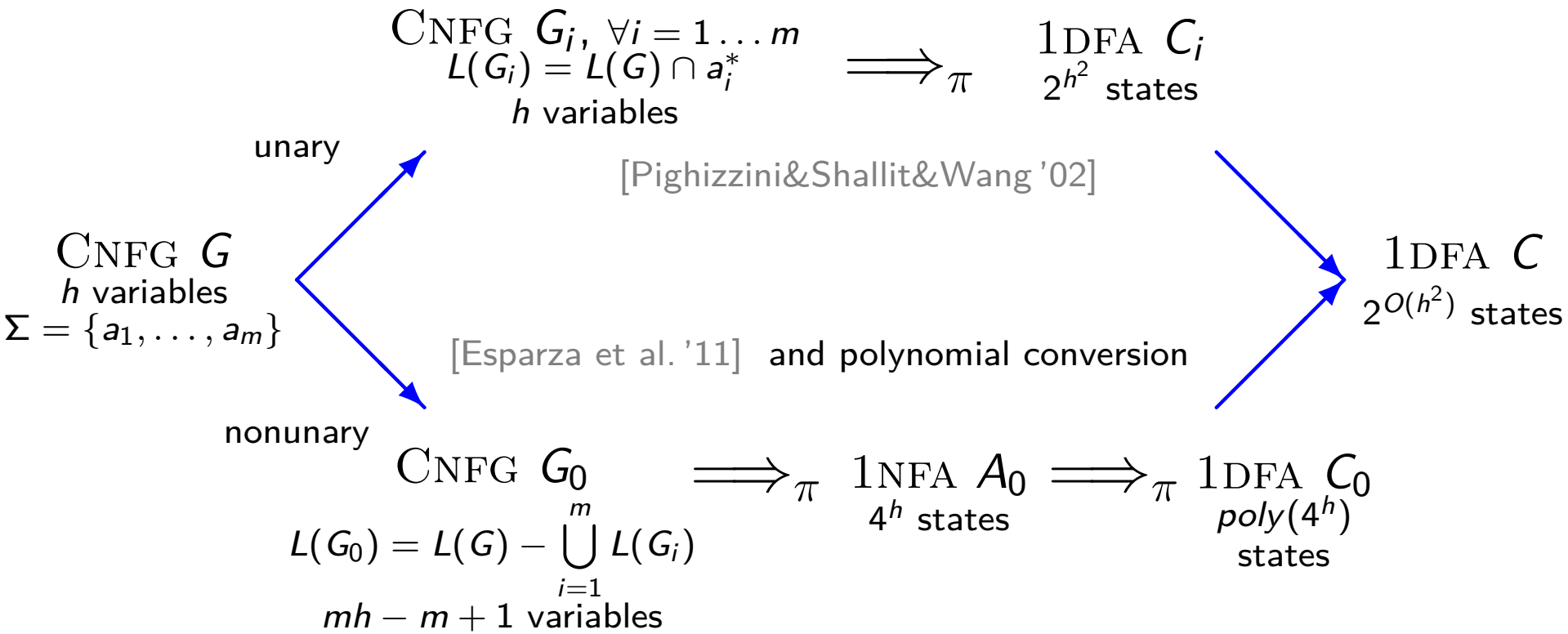
Our second contribution:

Problem (CFGs to 1DFAs)

$$\underset{\substack{\text{CFG } \textit{in Chomsky normal form}\\ h \textit{ variables}\\ L_1}}{\text{CNFG}} \quad \Longrightarrow_{\pi} \quad \underset{\substack{\textit{how many states?}\\ L_2}}{\text{1DFA}}$$

- Upper bound: $2^{O(4^h)}$
  by subset construction
  and [Esparza&Ganty&Kiefer&Luttenberger '11]
- Lower bound: 2^{ch^2}
  This bound derives from the *unary case*: the state cost of the conversion of unary h-variable CNFGs into equivalent 1DFAs is less than 2^{h^2} [Pighizzini&Shallit&Wang '02]

From CFGs to Parikh equivalent 1DFAs: proof idea



Theorem

For each h-variable CNFG *over Σ, there exists a Parikh equivalent* 1DFA *with at most $2^{O(h^2)}$ states. Furthermore, this cost is optimal.*

Conversions into Parikh equivalent 2DFAs

Summary table

Summarizing, we have the following optimal bounds:

	1DFA	2DFA
1NFA n states	$e^{O(\sqrt{n \cdot \ln n})}$	
CNFG h variables	$2^{O(h^2)}$	

Conversions into Parikh equivalent 2DFAs

Summary table

Our third contribution

	1DFA	2DFA
1NFA n states	$e^{O(\sqrt{n \cdot \ln n})}$	$poly(n)$
CNFG h variables	$2^{O(h^2)}$	$2^{O(h)}$

The idea is to split the accepted language into its unary and nonunary parts:

- The state cost of the conversion of unary n-state 1NFAs into equivalent 2DFAs is $O(n^2)$ [Chrobak '86]
- The state cost of the conversion of unary h-variable CNFGs into equivalent 1NFAs is at most $2^{2h-1} + 1$ [Pighizzini&Shallit&Wang '02]

Operational state complexity under Parikh equivalence

Our Goal

We investigate, under Parikh equivalence, the state complexity of some language operations which preserve regularity $(\cup, \cap, ^c, \cdot, ^*, \sqcup, ^R, P_{\Sigma_0})$.

Problem (1DFAs to 1DFA)

$$\begin{array}{c} A,\ B\ \text{1DFAs} \\ n_1,\ n_2\ \text{states} \\ L(A), L(B) \end{array} \quad \Longrightarrow_{\pi} \quad \begin{array}{c} C\ \text{1DFA} \\ L(C) =_{\pi} L \\ \textit{how many states?} \end{array}$$

where:

- $L = L(A) \cup L(B)$
- $L = L(A) \cap L(B)$
- $L = L(A)L(B)$
- ...

Concatenation under Parikh equivalence

Problem setting

Problem (1DFAs to 1DFA)

A, B 1DFAs		C 1DFA
n_1, n_2 states	$\Longrightarrow_\pi$	$L(C) =_\pi L$
$L = L(A)L(B)$		*how many states?*

- Upper bound: $e^{\sqrt{n \cdot \ln n}}$, where $n = n_1 + n_2$
  by Parikh equivalent conversion
- Lower bound: $n_1 n_2$ states
  by unary case

[Yu '00]

Concatenation under Parikh equivalence: proof idea

1DFAs A, B
n_1, n_2 states
$L = L(A)L(B)$
$\Sigma = \{a_1, \ldots, a_m\}$

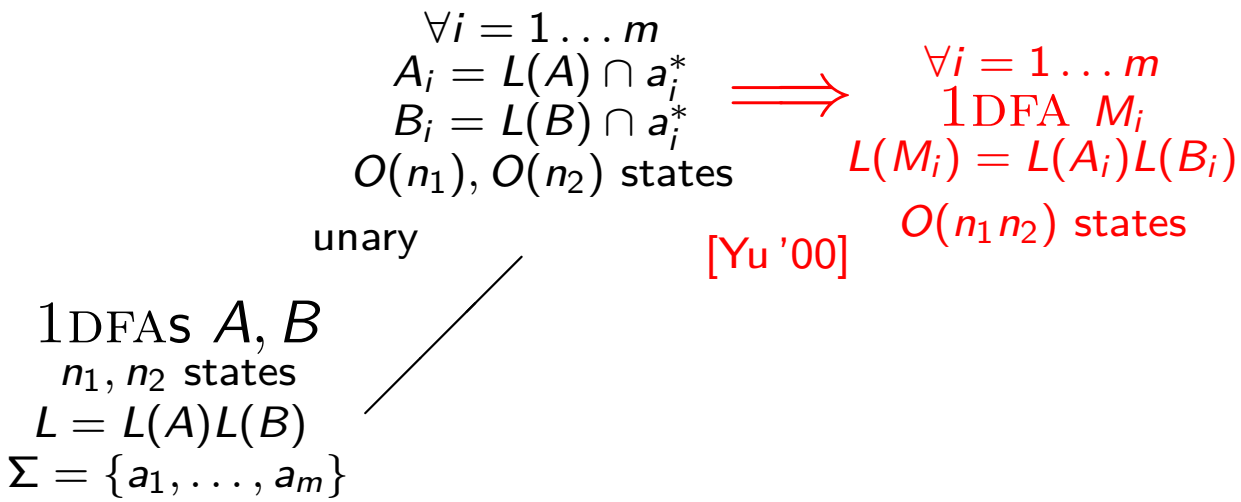
Concatenation under Parikh equivalence: proof idea

$\forall i = 1 \ldots m$
$A_i = L(A) \cap a_i^*$
$B_i = L(B) \cap a_i^*$
$O(n_1), O(n_2)$ states

unary

1DFAs A, B
n_1, n_2 states
$L = L(A)L(B)$
$\Sigma = \{a_1, \ldots, a_m\}$

Concatenation under Parikh equivalence: proof idea



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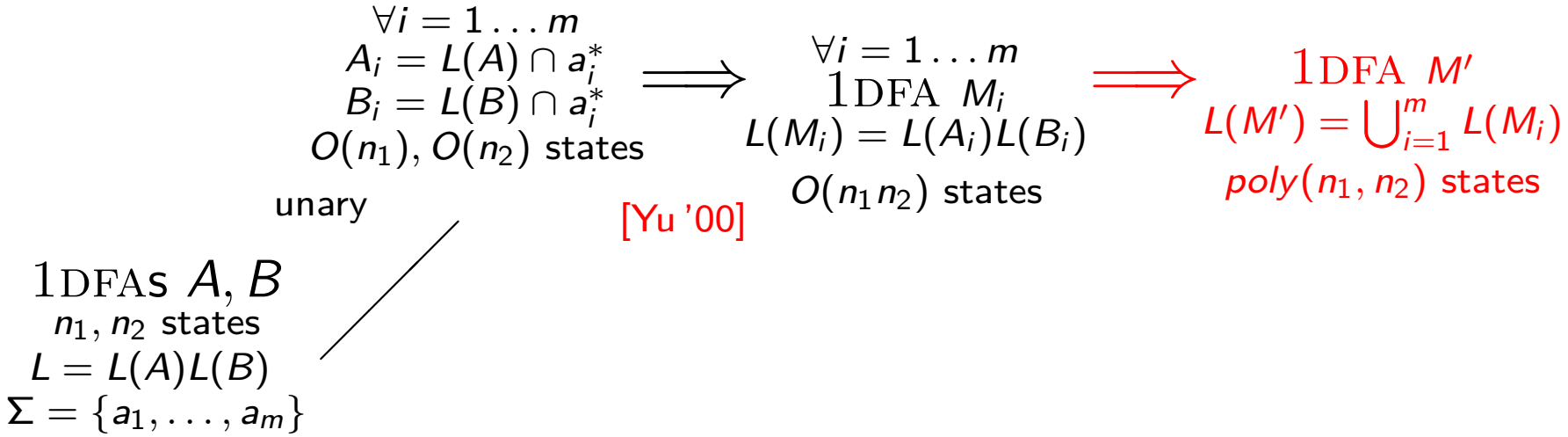

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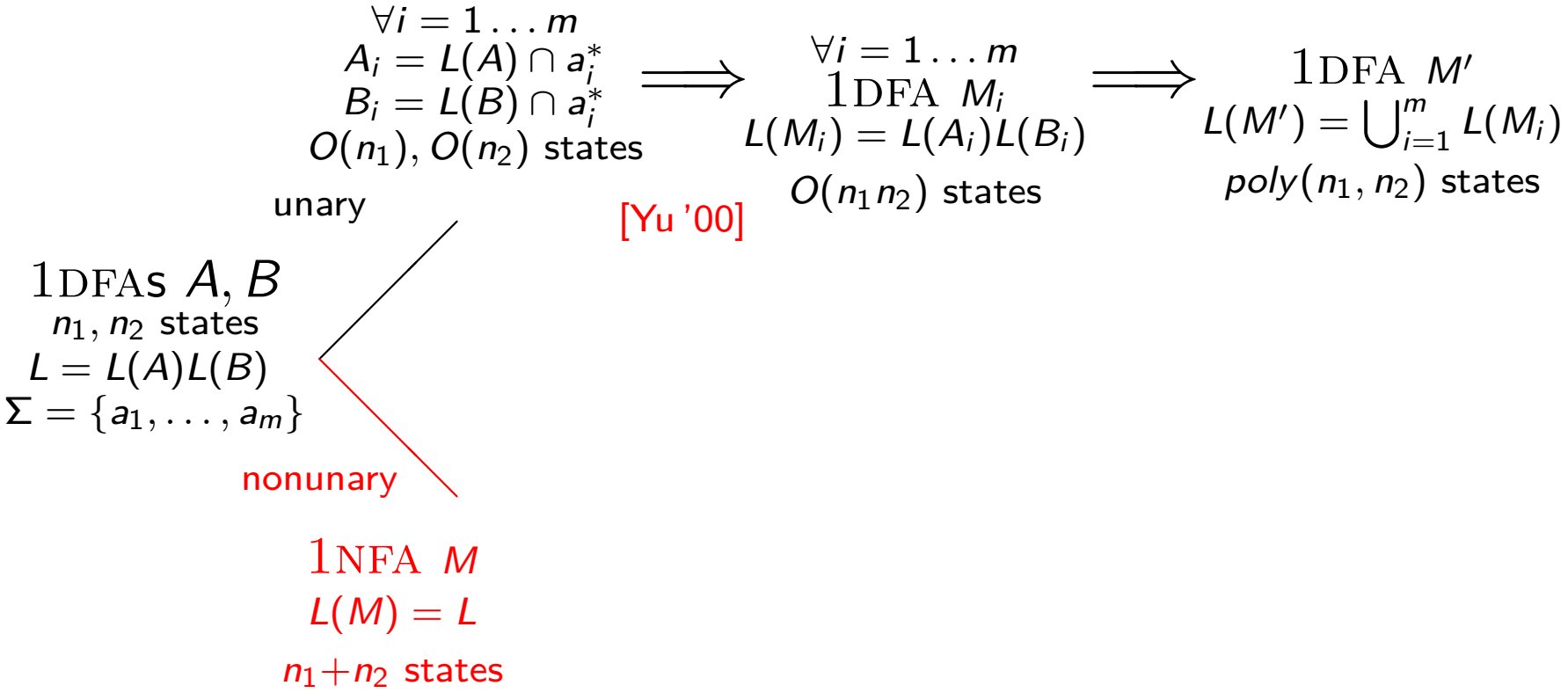

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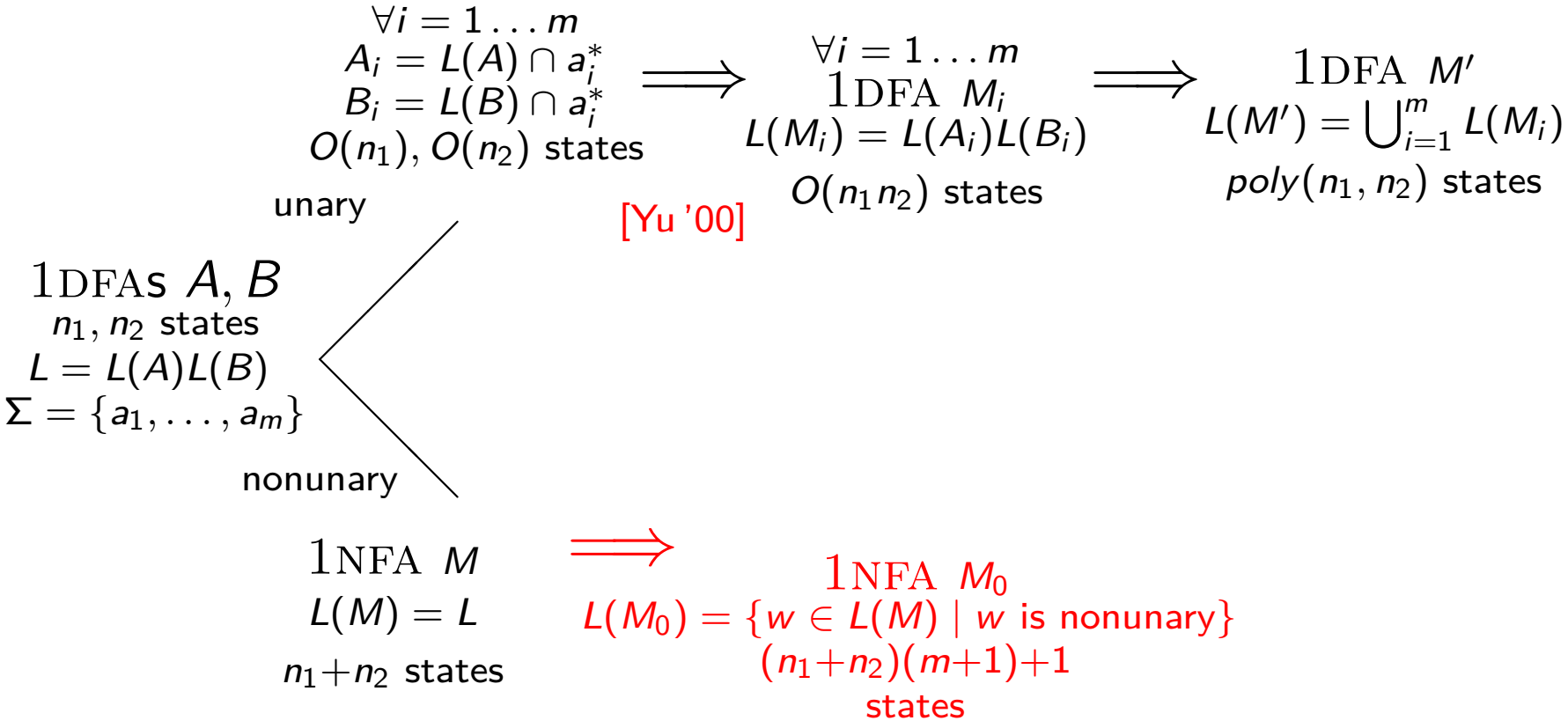

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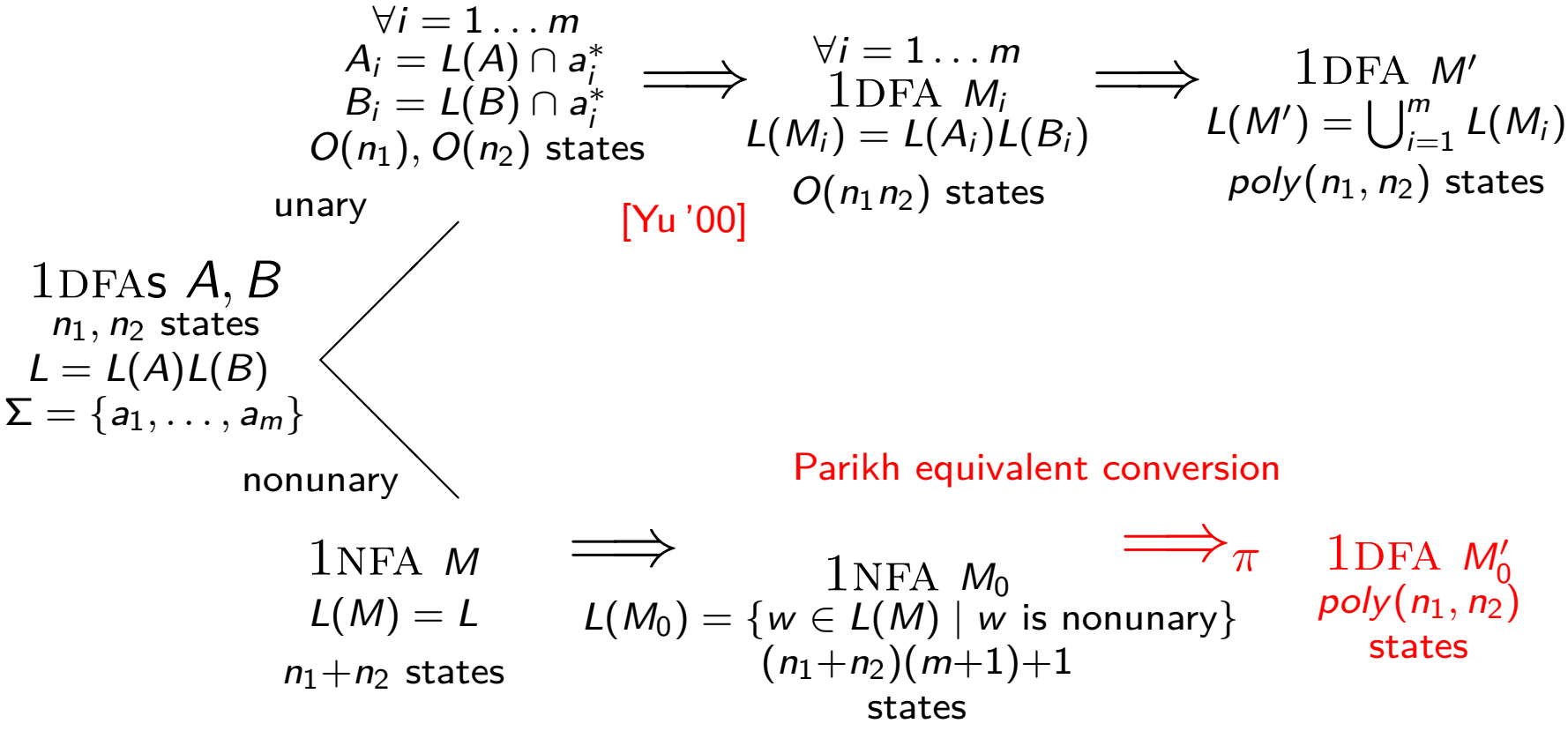

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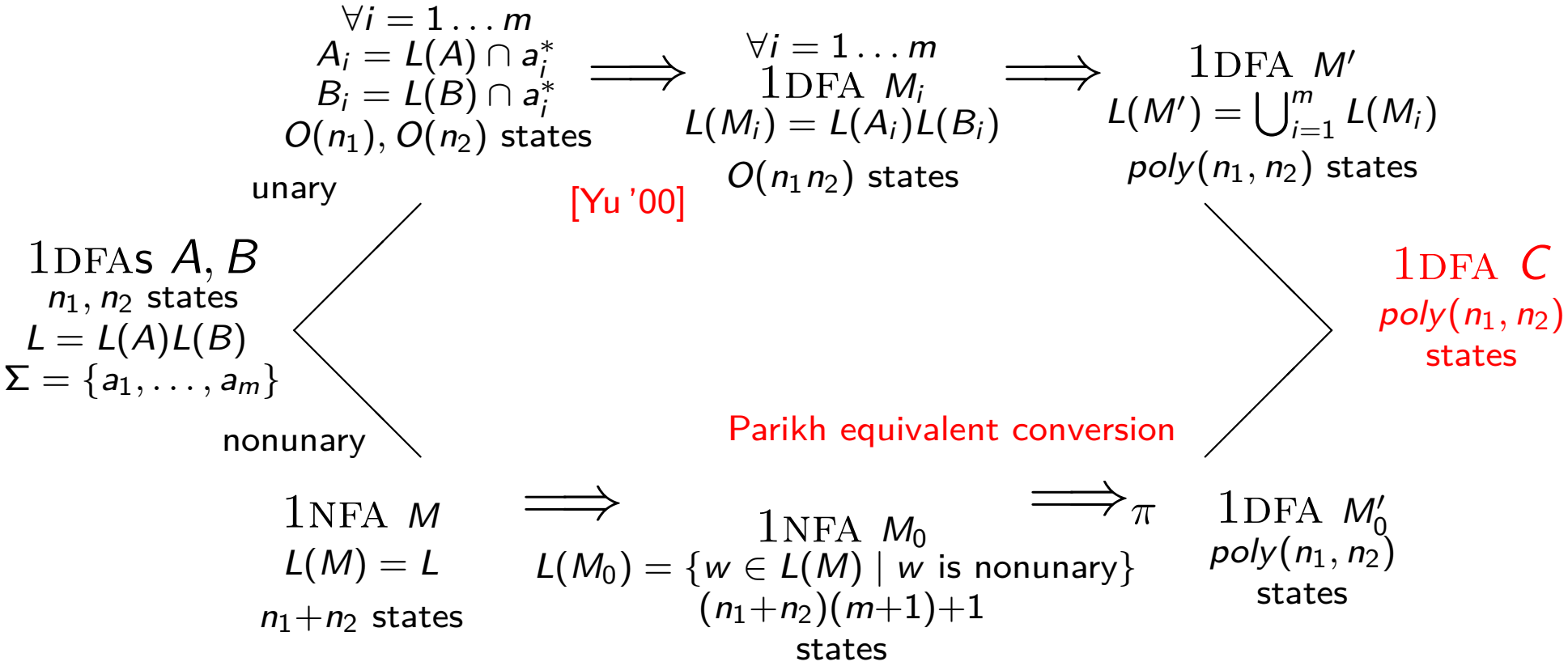

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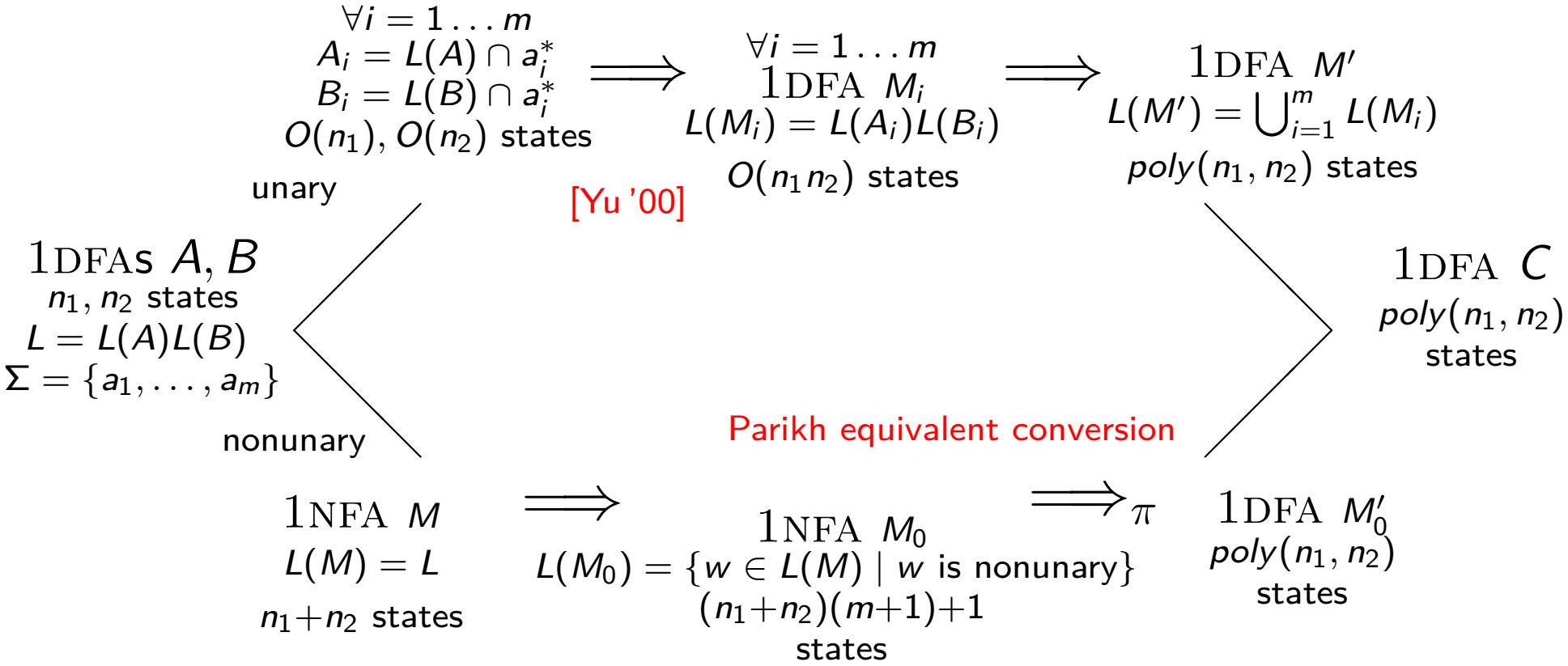

Theorem

Given two 1DFAs *A and B of* n_1 *and* n_2 *states, respectively, there exists a* 1DFA *of polynomial number of states in* n_1 *and* n_2 *that is Parikh equivalent to* $L(A)L(B)$*. Moreover, this cost is tight.*

Operational state complexity under Parikh equivalence

Summary table

Our fourth contribution:

Operation	Standard equivalence	Parikh equivalence
$L_1 \cup L_2$	$n_1 n_2$	$n_1 n_2$
$L_1 \cap L_2$	$n_1 n_2$	$n_1 n_2$
L_1^c	n_1	n_1
$L_1 L_2$	$(2n_1 - 1)2^{n_2 - 1}$	$poly(n_1, n_2)$
L_1^*	$2^{n_1 - 1} + 2^{n_1 - 2}$	$poly(n_1)$
$L_1 \sqcup\!\sqcup L_2$	$2^{n_1 n_2} - 1$	$poly(n_1, n_2)$
L_1^R	2^{n_1}	n_1
$P_{\Sigma_0}(L_1)$	$3 \cdot 2^{n_1 - 2} - 1$	$e^{O(\sqrt{n_1 \cdot \ln n_1})}$

[Yu '00, Campeanu&Salomaa&Yu '02, Yu&Zhuang&Salomaa '94, Jiraskova&Masopust '12]

Intersection: revisited

Non-commutativity with Parikh mapping

Intersection does not commute with Parikh mapping

$\psi(a^+b^+ \cap b^+a^+) \neq \psi(a^+b^+) \cap \psi(b^+a^+)$ holds; in fact,

$$\begin{aligned} \psi(a^+b^+ \cap b^+a^+) &= \emptyset \\ \psi(a^+b^+) \cap \psi(b^+a^+) &= \{(i,j) \mid i,j \geq 1\}. \end{aligned}$$

Problem: intersection

A, B 1DFAs
n_1, n_2 states

$\Longrightarrow$

M 1DFA
$\psi(L(M)) = \psi(L(A)) \cap \psi(L(B))$
How many states needed?

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Intersection: revisited

Theorem

Let A, B be 1DFA*s with respectively n_1, n_2 states over $\Sigma = \{a_1, \ldots, a_m\}$. There exists a* 1DFA *$M$ whose Parikh map is equal to $\psi(L(A)) \cap \psi(L(B))$ and which contains*

$$O(n^{(2m-1)(3m^3+6m^2)+2} p(n)^{2(3m^3+6m^2)+m})$$

states, where:

- $n = \max\{n_1, n_2\}(m+1)+1$
- $p(n) = O(n^{3m^2} m^{m^2/2+2})$

Proof.

Revisiting the Ginsburg and Spanier's proof [Ginsburg&Spanier '64] of the closure property of semilinear sets under intersection. □

Conclusion and future works

- Quite surprisingly, the costs into Parikh equivalent 1DFA is due to the unary parts of the languages.
- Conversion of the parts consisting of nonunary strings is less expensive.
- Conversions into Parikh equivalent 2DFA are less expensive than the corresponding ones into 1DFA.
- State complexity of some operations on regular languages have been studied under Parikh equivalence.
- For each two 1DFAs A and B, it is possible to obtain a 1DFA with a polynomial number of states, whose accepted language has as Parikh image $\psi(L(A)) \cap \psi(L(B))$.

Conclusion and future works

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Complement does not commute with Parikh mapping

$\psi((a^*b^*)^c) \neq (\psi(a^*b^*))^c$ holds; in fact,

$$\begin{aligned} \psi((a^*b^*)^c) &= \{(i,j) \mid i,j \geq 1\} \\ (\psi(a^*b^*))^c &= \emptyset. \end{aligned}$$

Problem: complement (left open!)

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n states

$\Longrightarrow$

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How many states needed?

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Given a deterministic automaton, finding the smallest Parikh equivalent deterministic automaton is an open question.

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